Math 2283 - Introduction to Logic

Quiz #10 - 2008.11.14

Solutions

Given the following axiom:
Ax IV: If $x < y$ and $y < z$, then $x < z$

Prove the following theorem:
Th: If $w < x$, $x < y$ and $y < z$, then $w < z$.

Hints: The following laws of logic may prove of some use.
Law I: $(P \wedge Q) \rightarrow Q$
Law II: $(P \wedge Q) \rightarrow P$
Law III: $P \rightarrow (Q \rightarrow (P \wedge Q))$

First we assume the antecedent of the theorem:

$$(w < x) \wedge (x < y) \wedge (y < z) \tag{1}$$

and attempt to show the consequent $w < z$.

By setting $P = (w < x)$ and $Q = (x < y) \wedge (y < z)$ in Laws I and II, we get

$$[(w < x) \wedge [(x < y) \wedge (y < z)]] \rightarrow [(x < y) \wedge (y < z)] \tag{2}$$

and

$$[(w < x) \wedge [(x < y) \wedge (y < z)]] \rightarrow (w < x) \tag{3}$$

respectively.

By (1) and (2) and L.O.D. we get

$$(x < y) \wedge (y < z). \tag{4}$$

Similarly, by (1) and (3) and L.O.D. we get

$$(w < x). \tag{5}$$

Notice now that by Ax IV and (4) with the L.O.D. we now also get

$$(x < z). \tag{6}$$

Using (5) and (6) as P and Q respectively in Law III gives

$$(w < x) \rightarrow ((x < z) \rightarrow ((w < x) \wedge (x < z))). \tag{7}$$

By L.O.D. with (5) and (7) we get

$$(x < z) \rightarrow ((w < x) \wedge (x < z)). \tag{8}$$

By L.O.D. with (6) and (8) we get

$$(w < x) \wedge (x < z). \tag{9}$$

Upon setting $x = w$ and $y = x$ in Ax IV we get

$$[(w < x) \wedge (x < z)] \rightarrow (w < z). \tag{10}$$

We are almost done now! Notice that (9) is the antecedent of (10), so applying L.O.D. gives $(w < z)$ which is exactly what we were trying to prove.