Math 2315 - Calculus II

Homework #3 - 2007.09.02

Due Date - 2007.09.12

Solutions

Part 1: Problems from sections 7.5.
Section 7.5:

31.

$$\int \sqrt{\frac{1+x}{1-x}}dx$$

First, we do some algebraic manipulation:

$$\begin{aligned}\int \sqrt{\frac{1+x}{1-x}}dx &= \int \sqrt{\frac{(1+x)(1+x)}{(1-x)(1+x)}}dx \\ &= \int \frac{1+x}{\sqrt{1-x^2}}dx \\ &= \int \frac{1}{\sqrt{1-x^2}}dx + \int \frac{x}{\sqrt{1-x^2}}dx \\ &= \sin^{-1}(x) - \sqrt{1-x^2} + C.\end{aligned}$$

65.

$$\int \frac{1}{\sqrt{x+1}+\sqrt{x}}dx$$

This one looks complicated, but is rather simple if you multiply top and bottom by the quantity $\sqrt{x+1}-\sqrt{x}$. This gives

$$\int \frac{1}{\sqrt{x+1}+\sqrt{x}}dx = \int \sqrt{x+1}-\sqrt{x}dx.$$

This integral is very straightforward, so we get

$$\int \frac{1}{\sqrt{x+1}+\sqrt{x}}dx = \frac{2}{3}(x+1)^{\frac{3}{2}} - \frac{2}{3}x^{\frac{3}{2}} + G.$$

Part 2: The *fun* problems.

1. Solve the following two hyperbolic integrals using a similar method to trigonometric integration.

a)

$$\int \sinh^4(x)\cosh^5(x)dx$$

We first remember that $\cosh^2(x) = 1 + \sinh^2(x)$. Since $\cosh(x)$ is raised to an odd power we rewrite the integral as

$$\int \sinh^4(x)\cosh^5(x)dx = \int \sinh^4(x)\left(\sinh^2(x)+1\right)^2\cosh(x)dx.$$

Now we let $u = \sinh(x)$ and $du = \cosh(x)dx$. This gives

$$\begin{aligned}\int \sinh^4(x)\left(\sinh^2(x)+1\right)^2\cosh(x)dx &= \int u^4\left(u^2+1\right)^2 du\\ &= \int u^8 + 2u^6 + u^4 du\\ &= \frac{1}{9}u^9 + \frac{2}{7}u^7 + \frac{1}{5}u^5 + S.\end{aligned}$$

Substituting back $u = \sinh(x)$, our final answer is

$$\int \sinh^4(x)\cosh^5(x)dx = \frac{1}{9}\sinh^9(x) + \frac{2}{7}\sinh^7(x) + \frac{1}{5}\sinh^5(x) + S.$$

b)

$$\int \cosh^2(x)dx$$

Here we use the identity $\cosh^2(x) = \frac{1}{2}(\cosh(2x)+1)$ to get

$$\begin{aligned}\int \cosh^2(x)dx &= \frac{1}{2}\int \cosh(2x) + 1dx\\ &\frac{1}{2}\left(\frac{1}{2}\sinh(2x) + x\right) + R.\end{aligned}$$

2. Suppose that $Q(x) = (x-a)(x-b)$, where $a \neq b$ and let $\frac{P(x)}{Q(x)}$ be a proper rational function so that

$$\frac{P(x)}{Q(x)} = \frac{A}{x-a} + \frac{B}{x-b}.$$

Show that

$$A = \frac{P(a)}{Q'(a)} \text{ and } B = \frac{P(b)}{Q'(b)}.$$

First, let us do some side calculations:

$$Q'(x) = (x-b) + (x-a), \quad Q'(a) = a-b, \quad Q'(b) = b-a.$$

Now, we start with the equation

$$\frac{P(x)}{Q(x)} = \frac{A}{x-a} + \frac{B}{x-b},$$

and take a derivative on both sides. This gives

$$\frac{P'(x)Q(x) - P(x)Q'(x)}{Q^2(x)} = -\frac{A}{(x-a)^2} - \frac{B}{(x-b)^2}.$$

Multiplying both sides by -1 gives

$$\frac{P(x)Q'(x) - P'(x)Q(x)}{Q^2(x)} = \frac{A}{(x-a)^2} + \frac{B}{(x-b)^2}.$$

Next, we multiply both sides by $Q^2(x)$:

$$P(x)Q'(x) - P'(x)Q(x) = Q^2(x)\left(\frac{A}{(x-a)^2} + \frac{B}{(x-b)^2}\right).$$

Now, we will plug in $x = a$. It is important to remember that $Q(a) = (a-a)(a-b)$. So we have

$$P(a)Q'(a) - P'(a)Q(a) = Q^2(a)\frac{A}{(a-a)^2} + Q^2(a)\frac{B}{(a-b)^2},$$

and the second term on the left goes to zero, as does the second term on the right. So we now have

$$P(a)Q'(a) = Q^2(a)\frac{A}{(a-a)^2}.$$

Here, we simplify the right hand side some more.

$$Q^2(a)\frac{A}{(a-a)^2} = (a-a)^2(a-b)^2\frac{A}{(a-a)^2} = A(a-b)^2.$$

Solving for A gives

$$\frac{P(a)Q'(a)}{(a-b)^2} = A.$$

However, $Q'(a) = (a-b)$, so $(a-b)^2 = (Q'(a))^2$. This implies that

$$\frac{P(a)Q'(a)}{(Q'(a))^2} = A,$$

and solving for A gives

$$A = \frac{P(a)}{Q'(a)}.$$

Now, we will plug in $x = b$. It is important to remember that $Q(b) = (b-a)(b-b)$. So we have

$$P(b)Q'(b) - P'(b)Q(b) = Q^2(b)\frac{A}{(b-a)^2} + Q^2(b)\frac{B}{(b-b)^2},$$

and the second term on the left goes to zero, as does the first term on the right. So we now have

$$P(b)Q'(b) = Q^2(b)\frac{B}{(b-b)^2}.$$

Here, we simplify the right hand side some more.

$$Q^2(b)\frac{B}{(b-b)^2} = (b-a)^2(b-b)^2\frac{B}{(b-b)^2} = B(b-a)^2.$$

Solving for B gives

$$\frac{P(b)Q'(b)}{(b-a)^2} = B.$$

However, $Q'(b) = (b-a)$, so $(b-a)^2 = (Q'(b))^2$. This implies that

$$\frac{P(b)Q'(b)}{(Q'(b))^2} = B,$$

and solving for B gives

$$B = \frac{P(b)}{Q'(b)}.$$

3. Let us now try to generalize problem 2 a little bit. Suppose that

$$Q(x) = (x-a_1)(x-a_2)\cdots(x-a_n) = \prod_{k=1}^{n}(x-a_k),$$

where $a_i \neq a_j$ for $1 \leq i, j, \leq n$. Let $\frac{P(x)}{Q(x)}$ be a proper rational function so that

$$\frac{P(x)}{Q(x)} = \frac{A_1}{x-a_1} + \frac{A_2}{x-a_2} + \cdots + \frac{A_n}{x-a_n} = \sum_{k=1}^{n}\frac{A_k}{x-a_k}.$$

The goal of this problem is to show that $A_j = \frac{P(a_j)}{Q'(a_j)}$. To make this a little easier, we will break it up a little bit. For the rest of this problem, we will thus assume that $Q(x) = \prod_{k=1}^{n}(x - a_k)$, as already stated.

a) Show that

$$Q'(x) = Q(x) \cdot \sum_{k=1}^{n} \frac{1}{x - a_k}.$$

So, first we start with

$$Q'(x) = \frac{d}{dx}\left((x - a_1)(x - a_2) \cdots (x - a_n)\right).$$

To compute the right hand side, we would have to use the product rule n times. For the kth term in the product rule derivative, the kth term vanishes. So we end up with something as follows:

$$\begin{aligned} Q'(x) =& (x - a_2)(x - a_3) \cdots (x - a_n) + (x - a_1)(x - a_3) \cdots (x - a_n) + \\ & \ldots + (x - a_1)(x - a_2) \cdots (x - a_{n-1}). \end{aligned}$$

Notice that the first term on the right can be rewritten as $\frac{Q(x)}{x-a_1}$, and following this pattern, we get

$$\begin{aligned} Q'(x) &= \frac{Q(x)}{x - a_1} + \frac{Q(x)}{x - a_2} + \ldots + \frac{Q(x)}{x - a_n} \\ &= Q(x)\left(\frac{1}{x - a_1} + \frac{1}{x - a_2} + \ldots + \frac{1}{x - a_n}\right) \\ &= Q(x) \cdot \sum_{k=1}^{n} \frac{1}{x - a_k}. \end{aligned}$$

b) Using part a) show that

$$Q'(a_j) = \prod_{k=1, k \neq j}^{n} (a_j - a_k)$$

So we start with the answer to part a) and let $x = a_j$.

$$Q'(a_j) = Q(a_j) \cdot \sum_{k=1}^{n} \frac{1}{a_j - a_k}.$$

Notice that we have

$$\frac{Q(a_j)}{a_j - a_k} = 0$$

if $j \neq k$. When $k = j$, the $a_j - a_j$ term in $Q(a_j)$ cancels the $a_j - a_j$ term in the numerator. Thus

$$\begin{aligned}
Q'(a_j) &= Q(a_j) \cdot \sum_{k=1, k\neq j}^{n} \frac{1}{a_j - a_k} + \frac{Q(a_j)}{a_j - a_j} \\
&= 0 + \frac{Q(a_j)}{a_j - a_j} \\
&= \left(\prod_{k=1}^{n} (a_j - a_k) \right) \frac{1}{a_j - a_j} \\
&= \prod_{k=1, k\neq j}^{n} (a_j - a_k).
\end{aligned}$$

c) From the definition of $Q(x)$ and parts a) and b), show that

$$Q^2(a_j) \cdot \sum_{k=1}^{n} \frac{A_k}{(a_j - a_k)^2} = \left(Q'(a_j)\right)^2 A_j.$$

We start with the left hand side, and show it is equal to the right hand side.

$$Q^2(a_j) \cdot \sum_{k=1}^{n} \frac{A_k}{(a_j - a_k)^2} = \prod_{r=1}^{n} (a_j - a_r)^2 \cdot \sum_{k=1}^{n} \frac{A_k}{(a_j - a_k)^2}$$

Now we expand the right hand side:

$$= \prod_{r=1}^{n} (a_j - a_r)^2 \cdot \left[\frac{A_1}{(a_j - a_1)^2} + \ldots + \frac{A_j}{(a_j - a_j)^2} + \ldots + \frac{A_n}{(a_j - a_n)^2} \right],$$

and then

$$= \left(\prod_{r=1, r\neq j}^{n} (a_j - a_k)^2 \right) \cdot (a_j - a_j)^2 \cdot \left[\frac{A_1}{(a_j - a_1)^2} + \ldots + \frac{A_j}{(a_j - a_j)^2} + \ldots + \frac{A_n}{(a_j - a_n)^2} \right],$$

$$= \left(\prod_{r=1, r\neq j}^{n} (a_j - a_k)^2 \right) \cdot \sum_{k=1}^{n} \frac{(a_j - a_j)^2}{(a_j - a_k)^2} A_k.$$

Now, if we take a closer look at that sum, we will notice that

$$\sum_{k=1}^{n} \frac{(a_j - a_j)^2}{(a_j - a_k)^2} A_k = A_j$$

since if $j = k$, the fraction becomes 1, otherwise it is 0. So now we have

$$\left(\prod_{r=1, r\neq j}^{n} (a_j - a_k)^2 \right) \cdot \sum_{k=1}^{n} \frac{(a_j - a_j)^2}{(a_j - a_k)^2} A_k = A_j \left(\prod_{r=1, r\neq j}^{n} (a_j - a_k)^2 \right).$$

Putting this together, we now have

$$Q^2(a_j) \cdot \sum_{k=1}^{n} \frac{A_k}{(a_j - a_k)^2} = A_j \left(\prod_{r=1, r\neq j}^{n} (a_j - a_k)^2 \right),$$

but the product on the left hand side is simply $(Q'(a_j))^2$ (simply square the equation form part b)). So now

$$Q^2(a_j) \cdot \sum_{k=1}^{n} \frac{A_k}{(a_j - a_k)^2} = \left(Q'(a_j)\right)^2 A_j.$$

d) Using parts a)-c) and your method of proving problem 2), show that

$$A_j = \frac{P(a_j)}{Q'(a_j)}.$$

So we start with

$$\frac{P(x)}{Q(x)} = \sum_{k=1}^{n} \frac{A_k}{x - a_k},$$

and take a derivative of both sides:

$$\frac{P'(x)Q(x) - P(x)Q'(x)}{Q^2(x)} = -\sum_{k=1}^{n} \frac{A_k}{(x - a_k)^2}.$$

We rewrite to get rid of the negative sign:

$$\frac{P(x)Q'(x) - P'(x)Q(x)}{Q^2(x)} = \sum_{k=1}^{n} \frac{A_k}{(x - a_k)^2},$$

and then multiply both sides by $Q^2(x)$:

$$P(x)Q'(x) - P'(x)Q(x) = Q^2(x) \sum_{k=1}^{n} \frac{A_k}{(x - a_k)^2}.$$

Next, we plug in $x = a_j$:

$$P(a_j)Q'(a_j) - P'(a_j)Q(a_j) = Q^2(a_j) \sum_{k=1}^{n} \frac{A_k}{(a_j - a_k)^2}.$$

By part c), the right hand side is simply $A_j(Q'(a_j))^2$, so

$$P(a_j)Q'(a_j) - P'(a_j)Q(a_j) = A_j(Q'(a_j))^2$$

and since $Q(a_j)$ for any j, we have

$$P(a_j)Q'(a_j) = A_j(Q'(a_j))^2.$$

Solving for A_j gives

$$A_j = \frac{P(a_j)}{Q'(a_j)}.$$