Math 2315 - Calculus II

Homework #18
Assigned - 2010.03.23
Due - 2010.03.29

Name: ______________________________

Textbook problems:

Section 9.8 - 7, 8, 10, 11, 12, 13, 16, 18, 23, 24, 33, 36

Fun Problems:

Do Exploratory Exercise 1, also find $P_3(x)$.

The n^{th} Legendre polynomial is defined, as stated in the book, by the formula

$$P_n(x) = 2^{-n} \sum_{k=0}^{[n/2]} \frac{(-1)^k (2n-2k)!}{(n-k)!k!(n-2k)!} x^{n-2k}.$$

So

$$P_0(x) = 2^{-0} \sum_{k=0}^{[0/2]} \frac{(-1)^k (0-2k)!}{(0-k)!k!(0-2k)!} x^{0-2k},$$

where $[0/2] = 0$. This gives

$$\begin{aligned} P_0(x) &= 2^{-0} \sum_{k=0}^{0} \frac{(-1)^k (0-2k)!}{(0-k)!k!(0-2k)!} x^{0-2k} \\ &= 2^{-0} \frac{(-1)^0\, 0!}{0!\,0!\,0!} x^0 \\ &= 1. \end{aligned}$$

$$P_1(x) = 2^{-1} \sum_{k=0}^{[1/2]} \frac{(-1)^k (2-2k)!}{(1-k)!k!(1-2k)!} x^{1-2k},$$

where $[1/2] = 0$. So

$$\begin{aligned} P_1(x) &= \frac{1}{2} \sum_{k=0}^{0} \frac{(-1)^k (2-2k)!}{(1-k)!k!(1-2k)!} x^{1-2k} \\ &= \frac{1}{2} \frac{(-1)^0\, 2!}{1!\,0!\,1!} x^{1-0} \\ &= x. \end{aligned}$$

Next up is $P_2(x)$:

$$P_2(x) = 2^{-2} \sum_{k=0}^{[2/2]} \frac{(-1)^k (4-2k)!}{(2-k)!k!(2-2k)!} \, x^{2-2k},$$

where $[2/2] = 1$. We will have two terms in this sum:

$$\begin{aligned} P_2(x) &= \frac{1}{2^2} \sum_{k=0}^{1} \frac{(-1)^k (4-2k)!}{(2-k)!k!(2-2k)!} \, x^{2-2k} \\ &= \frac{1}{2^2} \left[\frac{(1)\,4!}{2!\,0!\,2!} \, x^2 + \frac{(-1)\,2!}{1!\,1!\,0!} \, x^0 \right] \\ &= \frac{1}{4} \left[\frac{24}{4} \, x^2 - 2 \right] \\ &= \frac{3}{2} x^2 - \frac{1}{2}. \end{aligned}$$

Lastly, we get to $P_3(x)$, with $[3/2] = 1$, similar to $[2, 2]$.

$$\begin{aligned} P_3(x) &= 2^{-3} \sum_{k=0}^{1} \frac{(-1)^k (6-2k)!}{(3-k)!k!(3-2k)!} \, x^{3-2k} \\ &= \frac{1}{2^3} \left[\frac{(1)\,6!}{3!\,0!\,3!} \, x^3 + \frac{(-1)\,4!}{2!\,1!\,1!} \, x^1 \right] \\ &= \frac{1}{8} \left[20\, x^3 - 12\, x \right] \\ &= \frac{5}{2} x^2 - \frac{3}{2} \, x. \end{aligned}$$

It is a very straightforward exercise to show that $\int_{-1}^{1} P_n(x)\, P_m(x) = 0$ as the exercise asked. If we actually compute

$$\int_{-1}^{1} P_m(x)\, P_m(x)\, dx,$$

for $m = 0, 1, 2, 3$ we find

$$\begin{aligned} \int_{-1}^{1} P_0(x)\, P_0(x)\, dx &= 2 \\ \int_{-1}^{1} P_1(x)\, P_1(x)\, dx &= \frac{2}{3} \\ \int_{-1}^{1} P_2(x)\, P_2(x)\, dx &= \frac{2}{5} \\ \int_{-1}^{1} P_3(x)\, P_3(x)\, dx &= \frac{2}{7}, \end{aligned}$$

after which we can conjecture that for a non-negative integer m,

$$\int_{-1}^{1} P_m(x)\, P_m(x)\, dx = \frac{2}{2m+1}.$$

Next, we graph $P_k(x)$ for $k = 0, 1, 2, 3$ on the same graph.

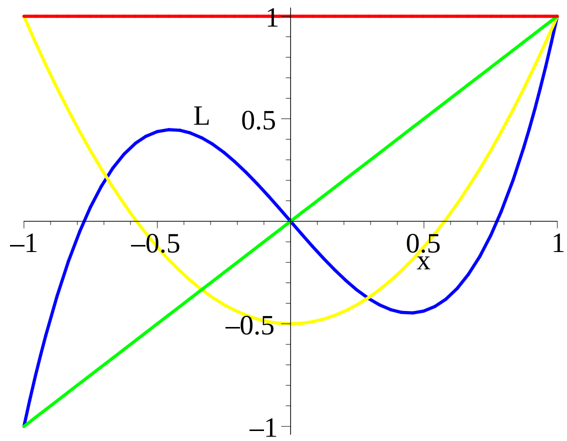

FIGURE 1. The first four Legendre Polynomials