Math 2315 - Calculus II

Homework #6 Solutions

Assigned - 2010.02.02

Due - 2010.02.08

Textbook problems:

Section 7.3 - 1, 2, 3, 8, 10, 13, 16, 20, 23, 26, 27, 31, 32, 36

Fun Problems:

1. Remember the identity $\cot^2(x) + 1 = \csc^2(x)$. Compute the following integrals:

a)

$$\int \cot^3(x)\csc(x)\,dx$$

We first do some algebra:

$$\begin{aligned}
\int \cot^3(x)\csc(x)\,dx &= \int \cot^2(x)\csc(x)\cot(x)\,dx \\
&= \int (\csc^2(x) - 1)\csc(x)\cot(x)\,dx
\end{aligned}$$

Now we can let $u = \csc(x)$, and therefore $du = -\csc(x)\cot(x)\,dx$.

$$\begin{aligned}
\int (\csc^2(x) - 1)\csc(x)\cot(x)\,dx &= \int (u^2 - 1)(-1)\,du \\
&= \int (1 - u^2)\,du \\
&= u - \frac{1}{3}u^3 + C \\
&= \csc(x) - \frac{1}{3}\csc^3(x) + C
\end{aligned}$$

b)

$$\int \cot^4(x)\csc^2(x)\,dx$$

This problem is a little more straight forward than part a). If we set $u = \cot(x)$, then $du = -\csc^2(x)\,dx$, and we have

$$\begin{aligned}
\int \cot^4(x)\csc^2(x)\,dx &= \int (u^4)(-1)\,du \\
&= -\int u^4\,du \\
&= -\frac{1}{5}u^5 + C \\
&= -\frac{1}{5}\cot^5(x) + C
\end{aligned}$$

2. Prove the reduction formula

$$\int \tan^k(x)\,dx = \frac{1}{k-1}\,\tan^{k-1}(x) - \int \tan^{k-2}(x)\,dx$$

Hint: $\tan^k(x) = (\sec^2(x) - 1)\tan^{k-2}(x)$.

We first use the hint:

$$\begin{aligned}\int \tan^k(x)\,dx &= \int (\sec^2(x) - 1)\tan^{k-2}(x)\,dx \\ &= \int \tan^{k-2}(x)\sec^2(x)\,dx - \int \tan^{k-2}(x)\,dx\end{aligned}$$

Then we work on the first integral on the left, with $u = \tan(x)$ and $du = \sec^2(x)\,dx$:

$$\begin{aligned}\int \tan^{k-2}(x)\sec^2(x)\,dx &= \int u^{k-2}\,du \\ &= \frac{1}{k-1}u^{k-1} \\ &= \frac{1}{k-1}\tan^{k-1}(x)\end{aligned}$$

The result given above can now be replaced back the first string of equalities to get

$$\int \tan^k(x)\,dx = \frac{1}{k-1}\,\tan^{k-1}(x) - \int \tan^{k-2}(x)\,dx$$

3. Prove that

$$\int \sin^2(x)\cos^3(x)\,dx = \frac{1}{30}\,(7 + 3\cos(2x))\sin^3(x) + C$$

First, we rewrite as

$$\begin{aligned}\int \sin^2(x)\cos^3(x)\,dx &= \int \sin^2(x)\cos^2(x)\cos(x)\,dx \\ &= \int \sin^2(x)(1 - \sin^2(x))\cos(x)\,dx,\end{aligned}$$

which tells us to use the substitution $u = \sin(x)$, since $du = \cos(x)\,dx$:

$$\begin{aligned}\int \sin^2(x)(1 - \sin^2(x))\cos(x)\,dx &= \int u^2(1 - u^2)\,du \\ &= \int u^2 - u^4\,du \\ &= \frac{1}{3}u^3 - \frac{1}{5}u^5 + C \\ &= \frac{1}{3}u^3\left(1 - \frac{3}{5}u^2\right) + C \\ &= \frac{1}{3}\sin^3(x)\left(1 - \frac{3}{5}\sin^2(x)\right) + C.\end{aligned}$$

We now use the face that $\sin^2(x) = \dfrac{1-\cos(2x)}{2}$:

$$\begin{aligned}
\frac{1}{3}\sin^3(x)\left(1 - \frac{3}{5}\sin^2(x)\right) + C &= \frac{1}{3}\sin^3(x)\left(1 - \frac{3}{5}\cdot\frac{1-\cos(2x)}{2}\right) + C \\
&= \frac{1}{3}\sin^3(x)\left(1 - \frac{3}{10} + \frac{3}{10}\cos(2x)\right) + C \\
&= \frac{1}{3}\sin^3(x)\left(\frac{7}{10} + \frac{3}{10}\cos(2x)\right) + C \\
&= \frac{1}{30}\sin^3(x)\left(7 + 3\cos(2x)\right) + C
\end{aligned}$$