Math 2013 - Introduction to Discrete Mathematics

Homework #4 - 2005.10.19

Due Date - 2005.10.28

Solutions

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1. Prove the following by induction:

$$\sum_{k=1}^{n} k^2 = \frac{1}{6}n(n+1)(2n+1)$$

Notice that when $n = 1$, the formula works. Next, we assume that it holds for n, and will show that it holds for $n+1$:

$$\begin{aligned}
\sum_{k=1}^{n+1} k^2 &= \sum_{k=1}^{n} k^2 + (n+1)^2 \\
&= \frac{1}{6}n(n+1)(2n+1) + (n+1)^2 \\
&= \frac{1}{6}(n+1)\left(n(2n+1) + 6(n+1)\right) \\
&= \frac{1}{6}(n+1)\left(2n^2 + 7n + 6\right) \\
&= \frac{1}{6}(n+1)(n+2)(2n+3) \\
\sum_{k=1}^{n+1} k^2 &= \frac{1}{6}(n+1)\left((n+1)+1\right)\left(2(n+1)+1\right)
\end{aligned}$$

2. Prove the following by induction:

$$\sum_{k=0}^{n} 2^k = 2^{n+1} - 1$$

Notice that when $n = 0$, the formula works. Next, we assume that it holds for n, and will show that it holds for $n+1$:

$$\begin{aligned}
\sum_{k=0}^{n+1} 2^k &= \sum_{k=0}^{n} 2^k + 2^{n+1} \\
&= 2^{k+1} - 1 + 2^{k+1} \\
&= 2 \cdot 2^{k+1} - 1 \\
\sum_{k=1}^{n+1} 2^k &= 2^{k+2} - 1
\end{aligned}$$

3. Show the following:

$$\sum_{k=1}^{2n} (-1)^k \cdot k = n$$

Hint: If you use the fact that

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2},$$

you will not have to use induction!

$$\begin{aligned}
\sum_{k=1}^{2n} (-1)^k \cdot k &= \sum_{k=1}^{n} (-1)^{2k} \cdot 2k + \sum_{k=1}^{n} (-1)^{2k-1} \cdot (2k-1) \\
&= \sum_{k=1}^{n} 2k - \sum_{k=1}^{n} (2k-1) \\
&= \sum_{k=1}^{n} 2k - \sum_{k=1}^{n} 2k + \sum_{k=1}^{n} 1 \\
\sum_{k=1}^{2n} (-1)^k \cdot k &= n(n+1) - n(n+1) + n = n
\end{aligned}$$

4. Given that $x_k \in \mathbb{R}$ for $k = 1, \ldots, n \geq 2$, prove the following by induction:

$$\left| \sum_{k=1}^{n} x_k \right| \leq \sum_{k=1}^{n} |x_k|$$

Hint: You might need the following facts: $-|a| \leq a \leq |a|$ and $|a| \leq b \Leftrightarrow -b \leq a \leq b$ for all a, $b \in \mathbb{R}$.

First we must show this for $n = 2$ I.e. is $|x_1 + x_2| \leq |x_1| + |x_2|$? We will use the above hints:

$$\begin{aligned}
-|x_1| \leq & x_1 \leq |x_1| \\
-|x_2| \leq & x_2 \leq |x_2|
\end{aligned}$$

Adding these gives:

$$-(|x_1| + |x_2|) \leq x_1 + x_2 \leq (|x_1| + |x_2|).$$

Using the fact that $|a| \leq b \Leftrightarrow -b \leq a \leq b$ gives the result for $n = 2$.

To show this is true for $n + 1$, we assume it is true for n. Thus we assume

$$\left| \sum_{k=1}^{n} x_k \right| \leq \sum_{k=1}^{n} |x_k|$$

and want to show

$$\left| \sum_{k=1}^{n+1} x_k \right| \leq \sum_{k=1}^{n} |x_k|.$$

To do this, note

$$\left| \sum_{k=1}^{n+1} x_k \right| = \left| \sum_{k=1}^{n} x_k + x_{n+1} \right|.$$

Setting

$$y = \sum_{k=1}^{n} x_k,$$

we have that

$$|y + x_{n+1}| \leq |y| + |x_{n+1}|$$

But we know that

$$\left|\sum_{k=1}^{n} x_k\right| \leq \sum_{k=1}^{n} |x_k|\,,$$

thus

$$\left|\sum_{k=1}^{n+1} x_k\right| \leq \sum_{k=1}^{n} |x_k|\,.$$