Math 2013 - Introduction to Discrete Mathematics

Exam #1 - 2015.09.24 Solutions

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1. Using the perturbation method, evaluate $\sum_{k=0}^{n} \frac{k}{2^k}$.

There are several ways to solve this, but we must use the perturbation method. If we define the partial sum S_n as follows:

$$S_n = \sum_{k=0}^{n} \frac{k}{2^k}$$

then

$$S_{n+1} = S_n + \frac{n+1}{2^{n+1}} = \sum_{k=0}^{n+1} \frac{k}{2^k}$$

Note that the first term on the right is zero, so

$$S_n + \frac{n+1}{2^{n+1}} = 0 + \sum_{k=1}^{n+1} \frac{k}{2^k}$$

Letting $k = k + 1$ in the right hand side of the above gives

$$\begin{aligned}
S_n + \frac{n+1}{2^{n+1}} &= \sum_{k=0}^{n} \frac{k+1}{2^{k+1}} \\
&= \sum_{k=0}^{n} \frac{k}{2^{k+1}} + \frac{1}{2^{k+1}} \\
&= \sum_{k=0}^{n} \frac{k}{2^{k+1}} + \sum_{k=0}^{n} \frac{1}{2^{k+1}} \\
&= \sum_{k=0}^{n} \frac{1}{2} \frac{k}{2^k} + \sum_{k=0}^{n} \frac{1}{2^{k+1}} \\
&= \frac{1}{2} \sum_{k=0}^{n} \frac{k}{2^k} + \sum_{k=0}^{n} \frac{1}{2^{k+1}} \\
S_n + \frac{n+1}{2^{n+1}} &= \frac{1}{2} S_n + \sum_{k=0}^{n} \frac{1}{2^{k+1}}
\end{aligned}$$

Solving for S_n in the last line above gives

$$\frac{1}{2} S_n = \sum_{k=0}^{n} \frac{1}{2^{k+1}} - \frac{n+1}{2^{n+1}}$$

Now, the sum on the right hand side is a geometric series, which has a closed form:

$$\sum_{k=0}^{n} \frac{1}{2^{k+1}} = 1 - \frac{1}{2^{n+1}}$$

Substituting this into the previous equation gives

$$\frac{1}{2} S_n = 1 - \frac{1}{2^{n+1}} - \frac{n+1}{2^{n+1}}$$

Solving for S_n gives (with a little algebraic manipulation):

$$S_n = 2 - \frac{n+2}{2^n}$$

2. Solve the following recurrence relation:

$$\begin{cases} T_0 &= 1 \\ T_n &= \frac{1}{n}T_{n-1} + \frac{1}{(n-1)!} \end{cases}$$

Following the method in Section 2.2, we set $a_n = 1$ and $b_n = \frac{1}{n}$ and look for the multiplicative factor s_n to allow for a substitution. Thus

$$\begin{aligned} s_n &= s_{n-1} \cdot \frac{a_{n-1}}{b_n} \\ &= n \cdot s_{n-1} \end{aligned}$$

Thus, since $s_n = n \cdot s_{n-1}$, we have $s_n = n!$. Multiplying both sides of the recurrence relation equation by s_n gives

$$n!\,T_n = (n-1)!\,T_{n-1} + n$$

Setting $S_n = n!\,T_n$, gives

$$\begin{cases} S_0 &= 1 \\ S_n &= S_{n-1} + n \end{cases}$$

Similarly, letting $n = n - 1$, we have

$$S_{n-1} = S_{n-2} + (n-1)$$

Thus

$$S_n = S_{n-2} + n + (n-1)$$

We can conclude then that

$$S_n = 1 + \sum_{k=1}^{n} k$$

The closed form for the sum of the first n integers is $\frac{n(n+1)}{2}$, so

$$S_n = 1 + \frac{n(n+1)}{2}$$

Substituting this in to the formula $S_n = n!\,T_n$ gives

$$T_n = \left(1 + \frac{n(n+1)}{2}\right) \cdot \frac{1}{n!}$$

3. Remember in your homework (and in class), we solved the system

$$\begin{cases} R_0 &= \alpha \\ R_n &= R_{n-1} + (-1)^n \left(\beta + n\gamma + n^2\delta\right) \end{cases}$$

With the solution being

$$R_n = A(n)\alpha + B(n)\beta + C(n)\gamma + D(n)\delta,$$

where

$$A(n) = 1, \quad B(n) = \frac{(-1)^n - 1}{2}, \quad C(n) = \frac{1}{2}(-1)^n n + \frac{1}{4}(-1)^n - \frac{1}{4}, \quad D(n) = (-1)^n \frac{n^2 + n}{2}$$

Use this to solve the summation $\displaystyle\sum_{k=0}^{n} (-1)^k \left(3 - 4k^2\right)$.

The first thing we do is convert to a recurrence relation:

$$\begin{cases} R_0 &= 3 \\ R_n &= R_{n-1} + (-1)^n \left(3 - 4n^2\right) \end{cases}$$

So $\alpha = \beta = 3$, $\gamma = 0$, and $\delta = -4$, and as a result,

$$R_n = 3 + 3\frac{(-1)^n - 1}{2} + -4(-1)^n \frac{n^2 + n}{2}.$$

4. Perform the following matrix multiplication:

$$\begin{bmatrix} 1 & 4 & 3 \\ 2 & 5 & 2 \\ 3 & 4 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & -1 \\ 1 & 0 & 2 \\ -2 & 1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 3 & 10 \\ 3 & 2 & 10 \\ 5 & 1 & 6 \end{bmatrix}$$