# Math 2215 - Calculus 1

## Exam #4 - 2017.11.13

## Solutions

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1. Using Riemann sums with right endpoints, compute the area between the curve and $x$-axis for $f(x) = x^3 - x^2 - 2x$ on the interval $[-1, 2]$.

So $x_0 = -1$, $x_n = 2$, $\Delta x = 3/n$, $x_k = x_0 + k\Delta x = -1 + \frac{3}{n}k$.

$$
\begin{aligned}
\mathcal{A}_n &= \sum_{k=1}^{n} f(x_k) \cdot \Delta x \\
&= \sum_{k=1}^{n} \left(x_k^3 - x_k^2 - 2x_k\right) \cdot \frac{3}{n} \\
&= \frac{3}{n} \sum_{k=1}^{n} \left[ \left(-1 + \frac{3}{n}k\right)^3 - \left(-1 + \frac{3}{n}k\right)^2 - 2\left(-1 + \frac{3}{n}k\right) \right] \\
&= \frac{3}{n} \sum_{k=1}^{n} \left[ \left(\frac{27}{n^3}k^3 - 3\frac{9}{n^2}k^2 + 3\frac{3}{n}k - 1\right) - \left(\frac{9}{n^2}k^2 - \frac{6}{n}k + 1\right) + 2 - \frac{6}{n}k \right] \\
&= \frac{3}{n} \sum_{k=1}^{n} \left(\frac{27}{n^3}k^3 - \frac{36}{n^2}k^2 + \frac{9}{n}k\right) \\
&= \frac{3}{n} \left[ \sum_{k=1}^{n} \frac{27}{n^3}k^3 - \sum_{k=1}^{n} \frac{36}{n^2}k^2 + \sum_{k=1}^{n} \frac{9}{n}k \right] \\
&= \frac{3}{n} \left[ \frac{27}{n^3} \sum_{k=1}^{n} k^3 - \frac{36}{n^2} \sum_{k=1}^{n} k^2 + \frac{9}{n} \sum_{k=1}^{n} k \right] \\
&= \frac{81}{n^4} \sum_{k=1}^{n} k^3 - \frac{108}{n^3} \sum_{k=1}^{n} k^2 + \frac{27}{n^2} \sum_{k=1}^{n} k \\
&= \frac{81}{n^4} \frac{n^2(n+1)^2}{4} - \frac{108}{n^3} \frac{n(n+1)(2n+1)}{6} + \frac{27}{n^2} \frac{n(n+1)}{2}
\end{aligned}
$$

Now we compute the limit as $n \to \infty$ of $\mathcal{A}_n$ since we have removed all summations:

$$
\begin{aligned}
\mathcal{A} &= \lim_{n\to\infty} \mathcal{A}_n \\
&= \lim_{n\to\infty} \left[ \frac{81}{n^4} \frac{n^2(n+1)^2}{4} - \frac{108}{n^3} \frac{n(n+1)(2n+1)}{6} + \frac{27}{n^2} \frac{n(n+1)}{2} \right] \\
&= \left[ \frac{81}{4} - \frac{108}{3} + \frac{27}{2} \right] \\
&= -\frac{9}{4}
\end{aligned}
$$

2. Use the Fundamental Theorem of Calculus to compute the area between the curve and $x$-axis for $f(x) = x^3 - x^2 - 2x$ on the interval $[-1, 2]$.

$$\begin{aligned}\mathcal{A} &= \int_{-1}^{2} x^3 - x^2 - 2x \, \mathrm{d}x \\ &= \frac{1}{4}x^4 - \frac{1}{3}x^3 - x^2 \Big|_{-1}^{2} \\ &= \left(4 - \frac{8}{3} - 4\right) - \left(\frac{1}{4} + \frac{1}{3} - 1\right) \\ &= -\frac{9}{4}\end{aligned}$$

3. Evaluate $\displaystyle\int x(x^2+1)^{1/4} \, \mathrm{d}x$.

We use the substitution $u = x^2 + 1$, and so $\mathrm{d}u = 2x\,\mathrm{d}x$. Solving for $x\,\mathrm{d}x$ gives $x\,\mathrm{d}x = \frac{1}{2}\,\mathrm{d}u$.

$$\begin{aligned}\int x(x^2+1)^{1/4} \, \mathrm{d}x &= \frac{1}{2}\int u^{1/4} \, \mathrm{d}u \\ &= \frac{1}{2} \cdot \frac{4}{5} u^{5/4} + \mathcal{C} \\ &= \frac{2}{5} \left(x^2+1\right)^{5/4} + \mathcal{C}\end{aligned}$$

4. Evaluate $\displaystyle\int x(x+1)^{1/4} \, \mathrm{d}x$.

We use the substitution $u = x + 1$, and so $\mathrm{d}u = \mathrm{d}x$. Solving for $x$ gives $x = u - 1$.

$$\begin{aligned}\int x(x+1)^{1/4} \, \mathrm{d}x &= \int (u-1)u^{1/4} \, \mathrm{d}u \\ &= \int u^{5/4} - u^{1/4} \, \mathrm{d}u \\ &= \frac{4}{9}u^{9/4} - \frac{4}{5}u^{5/4} + \mathcal{C} \\ &= \frac{4}{9}(x+1)^{9/4} - \frac{4}{5}(x+1)^{5/4} + \mathcal{C}\end{aligned}$$

5. Compute $\displaystyle\int_{-\pi}^{\pi/2} f(x) \, \mathrm{d}x$ if $f(x)$ is defined as follows:

$$f(x) = \begin{cases} 1 - \cos(x)\sin(x), & -\pi \le x < 0 \\ \sec^2(x/2), & 0 \le x \le \pi/2 \end{cases}$$

Since this is a piecewise defined function, we break up the definite integral as follows:

$$\int_{-\pi}^{\pi/2} f(x) \, \mathrm{d}x = \int_{-\pi}^{0} 1 - \cos(x)\sin(x) \, \mathrm{d}x + \int_{0}^{\pi/2} \sec^2(x/2) \, \mathrm{d}x$$

We will now work on each piece of the right-side of above. First

$$\int_{-\pi}^{0} 1 - \cos(x)\sin(x) \, \mathrm{d}x = \int_{-\pi}^{0} 1 \, \mathrm{d}x - \int_{-\pi}^{0} \cos(x)\sin(x) \, \mathrm{d}x.$$

The left integral on the right-hand side is simply $\pi$. So what remains is the right integral to compute. We use the substitution $u = \cos(x)$, then $\mathrm{d}u = -\sin(x)\,\mathrm{d}x$, and when $x = -\pi$, $u = -1$, and $x = 0$, $u = 1$. So

$$-\int_{-\pi}^{0} \cos(x)\sin(x) \, \mathrm{d}x = \int_{-1}^{1} u \, \mathrm{d}u = 0.$$

So

$$\int_{-\pi}^{0} 1-\cos(x)\sin(x)\,\mathrm{d}x = \pi + 0 = \pi.$$

Next, we work on the integral

$$\int_{0}^{\pi/2} \sec^2(x/2)\,\mathrm{d}x.$$

Setting $u = x/2$, then $\mathrm{d}u = \frac{1}{2}\,\mathrm{d}x$, or $2\,\mathrm{d}u = \mathrm{d}x$, with the limits $x = 0$ to $x = \pi/2$ becoming $u = 0$ to $u = \pi/4$.

$$\begin{aligned}\int_{0}^{\pi/2} \sec^2(x/2)\,\mathrm{d}x &= 2\int_{0}^{\pi/4} \sec^2(u)\,\mathrm{d}u \\ &= 2\,\tan(u)\Big|_{0}^{\pi/4} \\ &= 2\,(1-0) \\ &= 2\end{aligned}$$

Finally, we put all of this together to get

$$\begin{aligned}\int_{-\pi}^{\pi/2} f(x)\,\mathrm{d}x &= \int_{-\pi}^{0} 1-\cos(x)\sin(x)\,\mathrm{d}x + \int_{0}^{\pi/2} \sec^2(x/2)\,\mathrm{d}x \\ &= \pi + 2\end{aligned}$$

6. Evaluate $\displaystyle\int_{-4}^{4} x^3\cos^2(x^4) - x^2\sin(x^3)\,\mathrm{d}x$.

Note that $f(x) = x^3\cos^2(x^4) - x^2\sin(x^3)$ is odd since

$$\begin{aligned}f(-x) &= (-x)^3\cos^2((-x)^4) - (-x)^2\sin((-x)^3) \\ &= -x^3\cos^2(x^4) - x^2\cdot(-1)\cdot\sin(x^3) \\ &= -x^3\cos^2(x^4) + x^2\sin(x^3) \\ &= -f(x)\end{aligned}$$

Thus

$$\int_{-4}^{4} x^3\cos^2(x^4) - x^2\sin(x^3)\,\mathrm{d}x = 0.$$

7. Find limits of integration $a$ and $b$ which make the following equation true:

$$\int_{-1}^{2} g(x)\,\mathrm{d}x + \int_{0}^{4} g(x)\,\mathrm{d}x + \int_{2}^{0} g(x)\,\mathrm{d}x = \int_{a}^{b} g(x)\,\mathrm{d}x$$

$$\begin{aligned}\int_{-1}^{2} g(x)\,\mathrm{d}x + \int_{0}^{4} g(x)\,\mathrm{d}x + \int_{2}^{0} g(x)\,\mathrm{d}x &= \int_{-1}^{2} g(x)\,\mathrm{d}x + \int_{0}^{4} g(x)\,\mathrm{d}x - \int_{0}^{2} g(x)\,\mathrm{d}x \\ &= \int_{-1}^{2} g(x)\,\mathrm{d}x + \int_{0}^{2} g(x)\,\mathrm{d}x + \int_{2}^{4} g(x)\,\mathrm{d}x - \int_{0}^{2} g(x)\,\mathrm{d}x \\ &= \int_{-1}^{2} g(x)\,\mathrm{d}x + \int_{2}^{4} g(x)\,\mathrm{d}x \\ &= \int_{-1}^{4} g(x)\,\mathrm{d}x\end{aligned}$$

So $a = -1$ and $b = 4$ are the limits we are looking for.

8. Compute $F'(x)$ if

$$F(x) = \int_{\sqrt{2x+1}}^{\sin^2(x)} \cos^2(t) + \csc(t) - 1\,\mathrm{d}t$$

By the Fundamental Theorem of Calculus,

$$F(x) = G(\sin^2(x)) - G(\sqrt{2x+1}),$$

where $G'(t) = \cos^2(t) + \csc(t) - 1$. Thus

$$\begin{aligned}
F'(x) &= G'(\sin^2(x)) \cdot \frac{\mathrm{d}}{\mathrm{d}x} \sin^2(x) - G(\sqrt{2x+1}) \cdot \frac{\mathrm{d}}{\mathrm{d}x} \sqrt{2x+1} \\
&= \big(\cos^2(\sin^2(x)) + \csc(\sin^2(x)) - 1\big) \cdot \frac{\mathrm{d}}{\mathrm{d}x} \sin^2(x) \\
&\quad - \big(\cos^2(\sqrt{2x+1}) + \csc(\sqrt{2x+1}) - 1\big) \cdot \frac{\mathrm{d}}{\mathrm{d}x} \sqrt{2x+1} \\
&= \big(\cos^2(\sin^2(x)) + \csc(\sin^2(x)) - 1\big) \cdot 2\sin(x)\cos(x) \\
&\quad - \big(\cos^2(\sqrt{2x+1}) + \csc(\sqrt{2x+1}) - 1\big) \cdot \frac{2}{2\sqrt{2x+1}}
\end{aligned}$$