Math 2315 - Calculus 2

Cumulative Quiz #3 - 2021.03.12

Solutions

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1. Compute the following derivative: $\dfrac{\mathrm{d}}{\mathrm{d}x}\tan^{-1}(\sin(3x))$

   We use the chain rule here:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\tan^{-1}(\sin(3x)) &= \frac{1}{1+\sin^2(3x)}\cdot\frac{\mathrm{d}}{\mathrm{d}x}\sin(3x)\\ &= \frac{3\cos(3x)}{1+\sin^2(3x)}\end{aligned}$$

2. Compute the following integrals:

(a) $\displaystyle\int \frac{w^3}{1+w^8}\,\mathrm{d}w$

   We first note that $w^8 = \left(w^4\right)^2$, thus

$$\int \frac{w^3}{1+w^8}\,\mathrm{d}w = \int \frac{w^3}{1+\left(w^4\right)^2}\,\mathrm{d}w$$

Furthermore since the numerator has a w^3, which is almost the derivative of w^4, we perform the substitution $u = w^4$, with $\mathrm{d}u = 4w^3\mathrm{d}w$. Thus

$$\begin{aligned}\int \frac{w^3}{1+w^8}\,\mathrm{d}w &= \int \frac{w^3}{1+\left(w^4\right)^2}\,\mathrm{d}w\\ &= \frac{1}{4}\int\frac{1}{1+u^2}\,\mathrm{d}u\\ &= \frac{1}{4}\tan^{-1}(u)+\mathcal{C}\\ &= \frac{1}{4}\tan^{-1}\left(w^4\right)+\mathcal{C}\end{aligned}$$

(b) $\displaystyle\int \frac{1}{u^2(3+4u)}\,\mathrm{d}u$

   To do this integral, we do partial fractions:

$$\frac{1}{u^2(3+4u)} = \frac{A}{u}+\frac{B}{u^2}+\frac{C}{3+4u}$$

Multiplying through the entire equation by $u^2(3+4u)$ to get

$$1 = Au(3+4u)+B(3+4u)+Cu^2$$

Setting $u=0$ gives the equation $1=3B$, or $B=\frac{1}{3}$. Setting $u=-\frac{3}{4}$ gives $C=\frac{16}{9}$, and for the last unknown, A, we need to pick another value of u, so setting $u=1$, we end up with $A=-\frac{4}{9}$. So now we have

$$\begin{aligned}\int\frac{1}{u^2(3+4u)}\,\mathrm{d}u &= -\frac{4}{9}\int\frac{1}{u}\,\mathrm{d}u+\frac{1}{3}\int\frac{1}{u^2}\,\mathrm{d}u+\frac{16}{9}\int\frac{1}{3+4u}\,\mathrm{d}u\\ &= -\frac{4}{9}\ln(|u|)-\frac{1}{3}\frac{1}{u}+\frac{4}{9}\ln(|3+4u|)+\mathcal{C}\end{aligned}$$

3. Compute the following limit: $\displaystyle\lim_{x\to 0}\frac{\tan^{-1}(3x)}{\sin(2x)}$

   If we plug in $x=0$, we are of the form $\frac{0}{0}$ which means we can apply l'Hôpital's rule:

$$\lim_{x\to 0}\frac{\tan^{-1}(3x)}{\sin(2x)} = \lim_{x\to 0}\frac{\frac{\mathrm{d}}{\mathrm{d}x}\tan^{-1}(3x)}{\frac{\mathrm{d}}{\mathrm{d}x}\sin(2x)}$$
$$= \lim_{x\to 0}\frac{\frac{3}{1+9x^2}}{2\cos(2x)}$$
$$= \frac{3}{2}$$

4. Determine whether the following sequences converge or diverge:

(a) $a_n = \dfrac{2n^2\cos(n)}{n^2+4}$

We rewrite this as

$$a_n = \frac{2n^2}{n^2+4}\cdot\cos(n),$$

and note that as $n\to\infty$, the first term tends to 2, while the $\cos(n)$ term oscillates between -1 and 1. Thus the sequence diverges.

(b) $a_n = \dfrac{2n^2\cos\left((2n+1)\pi\right)}{n^2+4}$

We rewrite this as

$$a_n = \frac{2n^2}{n^2+4}\cdot\cos\left((2n+1)\pi\right),$$

and note that as $n\to\infty$, the first term tends to 2, while the $\cos\left((2n+1)\pi\right)$ term is always -1. Thus the sequence converges to -2.

(c) $a_n = \dfrac{2n^2\cos\left(1/n\right)}{n^2+4}$

We rewrite this as

$$a_n = \frac{2n^2}{n^2+4}\cdot\cos\left(1/n\right),$$

and note that as $n\to\infty$, the first term tends to 2, while the $\cos\left(1/n\right)$ term goes to $\cos(0)=1$ as $n\to\infty$. Thus, the limit exists, and it is 2.

5. Determine whether the following series converge or diverge:

(a) $\displaystyle\sum_{k=1}^{\infty}\frac{\cos(1/k)}{\sqrt{k^3+1}}$

We can use the comparison test after recognizing that $0\le\cos(1/k)\le 1$ for $1\le k<\infty$. Thus

$$0 < \frac{\cos(1/k)}{\sqrt{k^3+1}} < \frac{1}{\sqrt{k^3+1}} < \frac{1}{\sqrt{k^3}}$$

Since we know

$$\sum_{k=1}^{\infty}\frac{1}{k^{3/2}}$$

converges (p- series test with $p=3/2>1$), by the comparison test, the series converges.

(b) $\displaystyle\sum_{k=6}^{\infty}\frac{2k+3\cos(2k+1)}{k^3}$

This can be done several ways, but we will use the comparison test. We start with the following observation for all k:

$$-3\le 3\cos(2k+1)\le 3,$$

and therefore,

$$-3 + 2k \leq 2k + 3\cos(2k+1) \leq 3 + 2k,$$

which gives (at least for $k > 6$ which we are concerned with):

$$0 \leq \frac{2k + 3\cos(2k+1)}{k^3} \leq \frac{3+2k}{k^3}.$$

We now have

$$\sum_{k=6}^{\infty} \frac{2k + 3\cos(2k+1)}{k^3} \leq \sum_{k=6}^{\infty} \frac{2k+3}{k^3}.$$

We can show that

$$\sum_{k=6}^{\infty} \frac{2k+3}{k^3} < \infty$$

by using the limit comparison test with $a_k = \dfrac{2k+3}{k^3}$ and $b_k = \dfrac{1}{k^2}$. Therefore, the series is convergent.

6. Compute exactly: $\displaystyle\sum_{k=0}^{\infty} 3 \cdot \left(-\frac{2}{3}\right)^k$.

This is a geometric series with $r = -2/3$ and $a = 3$. Thus

$$\sum_{k=0}^{\infty} 3 \cdot \left(-\frac{2}{3}\right)^k = \frac{3}{1-(-2/3)} = \frac{9}{5}$$