Math 2315 - Calculus 2

Quiz #1 - 2022.01.21

Solutions

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1. Evaluate the following trigonometric functions at the given angles.
   (a) $\sin(0) = 0$
   (b) $\cos(\pi/2) = 0$
   (c) $\tan(\pi/4) = 1$
   (d) $\cos(\pi/6) = \sqrt{3}/2$

2. Evaluate the following inverse trigonometric functions at the given side length ratios.
   (a) $\cos^{-1}(1/2) = \pi/3$
   (b) $\sin^{-1}(-1/2) = -\pi/6$
   (c) $\tan^{-1}(-1) = -\pi/4$
   (d) $\sec^{-1}(2) = \pi/3$

3. Fill in the right hand side of the following exponential and logarithmic identities:
   (a) $e^a e^b = e^{a+b}$
   (b) $e^a/e^b = e^{a-b}$
   (c) $(e^a)^b = e^{ab}$
   (d) $\ln(ab) = \ln(a) + \ln(b)$
   (e) $\ln(a/b) = \ln(a) - \ln(b)$
   (f) $\ln(a^b) = b\ln(a)$

4. Compute the following derivatives:
   (a) $\frac{d}{\mathrm{d}x} \sin(x)e^{\cos(x)}$

   We employ the product rule here:

$$\begin{aligned}\frac{d}{\mathrm{d}x}\sin(x)e^{\cos(x)} &= \cos(x)e^{\cos(x)} + \sin(x)\cdot e^{\cos(x)}\cdot(-\sin(x)) \\ &= \cos(x)e^{\cos(x)} - \sin^2(x)e^{\cos(x)}\end{aligned}$$

   (b) $\frac{d}{\mathrm{d}x}\frac{(3x+1)^{15}\sin(3x+2)(4x^3-5x+2)^{2/5}}{\cos(7x+3)(4x-1)^{7/6}}$

   We will use logarithmic differentiation on this one. Setting the expression to $f(x)$, and recognizing that

$$\ln(f(x)) = 15\ln(3x+1) + \ln(\sin(3x+2)) + 2/5\ln(4x^3-5x+2) - \ln(\cos(7x+3)) - 7/6\ln(4x-1)$$

$$\begin{aligned}\frac{d}{\mathrm{d}x}f(x) &= f(x)\cdot\frac{d}{\mathrm{d}x}\ln(f(x)) \\ &= f(x)\cdot\left[15\frac{3}{3x+1} + \frac{3\cos(3x+2)}{\sin(3x+2)} + \frac{2}{5}\frac{12x^2-5}{4x^3-5x+2} - \frac{-7\sin(7x+3)}{\cos(7x+3)} - \frac{7}{6}\frac{4}{4x-1}\right]\end{aligned}$$

   (c) $\frac{d}{\mathrm{d}z}2^{1+\cos^2(z)}\log_3(e^{2z}+e^{-2z})$

$$\begin{aligned}\frac{d}{\mathrm{d}z}\left(2^{1+\cos^2(z)}\log_3(e^{2z}+e^{-2z})\right) &= \ln(2)\cdot 2^{1+\cos^2(z)}\cdot\frac{d}{\mathrm{d}z}\left(1+\cos^2(z)\right)\cdot\log_3(e^{2z}+e^{-2z}) \\ &\quad + 2^{1+\cos^2(z)}\frac{1}{\ln(3)}\frac{\frac{d}{\mathrm{d}z}(e^{2z}+e^{-2z})}{e^{2z}+e^{-2z}} \\ &= \ln(2)\cdot 2^{1+\cos^2(z)}\cdot(-2\cos(z)\sin(z))\cdot\log_3(e^{2z}+e^{-2z}) \\ &\quad + 2^{1+\cos^2(z)}\frac{1}{\ln(3)}\frac{2e^{2z}-2e^{-2z}}{e^{2z}+e^{-2z}}\end{aligned}$$

(d) $\frac{d}{dw}\log_2(\log_3(\ln(w)))$

$$\begin{aligned}\frac{d}{dw}\log_2(\log_3(\ln(w))) &= \frac{1}{\ln(2)}\frac{1}{\log_3(\ln(w))}\cdot\frac{d}{dw}\log_3(\ln(w)\\ &= \frac{1}{\ln(2)}\frac{1}{\log_3(\ln(w))}\cdot\frac{1}{\ln(3)}\frac{1}{\ln(w)}\cdot\frac{d}{dw}\ln(w)\\ &= \frac{1}{\ln(2)}\frac{1}{\log_3(\ln(w))}\cdot\frac{1}{\ln(3)}\frac{1}{\ln(w)}\cdot\frac{1}{w}\end{aligned}$$

5. If $f(x)=\sqrt[3]{5x^3-3x+6}$, compute the tangent line to $f^{-1}(x)$ at $x=2$.

First note that $f(1)=2$, so $f^{-1}(2)=1$. So the equation of the tangent line is $y-1=m(x-2)$, where $m=1/f'(f^{-1}(2))=1/f'(1)$. So we take a derivative of $f(x)$.

$$f'(x)=\frac{1}{3}(5x^3-3x+6)^{-2/3}\cdot(15x^2-3)=\frac{15x^2-3}{3(\sqrt[3]{5x^3-3x+6})^2}$$

Thus, $f'(1)=12/12=1$, so the equation of the tangent line is $y-1=1(x-2)$.

6. Compute the following integrals.

(a) $\int\frac{1+2w}{3w^2}\,\mathrm{d}w$

$$\begin{aligned}\int\frac{1+2w}{3w^2}\,\mathrm{d}w &= \int\frac{1}{3w^2}+\frac{2w}{3w^2}\,\mathrm{d}w\\ &= \int\frac{1}{3w^2}+\frac{2}{3}\frac{1}{w}\,\mathrm{d}w\\ &= -\frac{1}{3w}+\frac{2}{3}\ln(|w|)+\mathcal{C}\end{aligned}$$

(b) $\int\frac{-2z+1}{e^{z^2-z}}\,\mathrm{d}z$

We can use properties of exponentials to first move the denominator to the numerator. The we set $u=-z^2+z$ with $\mathrm{d}u=-2z+1\mathrm{d}z$ to finish the integral.

$$\begin{aligned}\int\frac{-2z+1}{e^{z^2-z}}\,\mathrm{d}z &= \int(-2z+1)e^{-z^2+z}\,\mathrm{d}z\\ &= \int e^u,\mathrm{d}u\\ &= e^u+\mathcal{C}\\ &= e^{-z^2+z}+\mathcal{C}\end{aligned}$$

(c) $\int_{\pi/2}^{\pi}2\cot(\theta/3)\,\mathrm{d}\theta$

After rewriting cotangent in terms of sine and cosine, we will then use the substitution $u=\sin(\theta/3)$, with $3\mathrm{d}u=\cos(\theta/3)\mathrm{d}\theta$. Note also that $u(\pi/2)=1/2$ and $u(\pi)=\sqrt{3}/2$.

$$\begin{aligned}\int_{\pi/2}^{\pi}2\cot(\theta/3)\,\mathrm{d}\theta &= 2\int_{\pi/2}^{\pi}\frac{\cos(\theta/3)}{\sin(\theta/3)}\,\mathrm{d}\theta\\ &= 6\int_{1/2}^{\sqrt{3}/2}\frac{1}{u}\,\mathrm{d}u\\ &= 6\ \ln(u)\big|_{1/2}^{\sqrt{3}/2}\\ &= 6\left(\ln(\sqrt{3}/2)-\ln(1/2)\right)\end{aligned}$$

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