Math 2315 - Calculus 2

Quiz #2 - 2022.02.07

Solutions

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1. Evaluate the following trigonometric functions at the given angles.
   (a) $\sin(2\pi/3) = \sqrt{3}/2$
   (b) $\cos(5\pi/6) = -\sqrt{3}/2$
   (c) $\cot(\pi/3) = 1/\sqrt{3}$
   (d) $\sec(-\pi/3) = 2$

2. Evaluate the following inverse trigonometric functions at the given side length ratios.
   (a) $\cos^{-1}(-1/2) = 2\pi/3$
   (b) $\sin^{-1}(-1/\sqrt{2}) = -\pi/4$
   (c) $\tan^{-1}(\sqrt{3}) = \pi/3$
   (d) $\csc^{-1}(-\sqrt{2}) = -\pi/4$

3. If $f(x) = 4x^5 + 2x + 1$, compute the tangent line to $f^{-1}(x)$ at $x = 1$.

First note that $f(0) = 1$, so $f^{-1}(1) = 0$. So the equation of the tangent line is $y - 0 = m(x - 1)$, where $m = 1/f'(f^{-1}(1)) = 1/f'(0)$. So we compute $f'(x) = 20x^4 + 2$. Thus, $f'(0) = 2$, so the equation of the tangent line is $y - 0 = 1/2(x - 1)$.

4. Compute the following derivatives:
   (a) $\frac{d}{\mathrm{d}x}\sqrt{x}\sinh(3x) + e^{2x}\cosh(4x)$

   We employ the product rule here twice:

$$\frac{d}{\mathrm{d}x}\sqrt{x}\sinh(3x) + e^{2x}\cosh(4x) = \frac{1}{2\sqrt{x}} \cdot \sinh(3x) + \sqrt{x} \cdot 3\cosh(3x) + 2e^{2x}\cosh(x) + e^{2x} \cdot 4\sinh(4x)$$

   (b) $\frac{d}{\mathrm{d}y}\log_6(3y^2 + 2)$

$$\frac{d}{\mathrm{d}y}\log_6(3y^2 + 2) = \frac{1}{\ln(6)} \cdot \frac{6y}{3y^2 + 2}$$

   (c) $\frac{d}{\mathrm{d}z}\tan^{-1}\left(\frac{1}{z^2 - 1}\right)$

$$\begin{aligned}\frac{d}{\mathrm{d}z}\tan^{-1}\left(\frac{1}{z^2 - 1}\right) &= \frac{1}{1 + \left(\frac{1}{z^2-1}\right)^2} \cdot \frac{d}{\mathrm{d}z}\frac{1}{z^2 - 1} \\ &= \frac{1}{1 + \left(\frac{1}{z^2-1}\right)^2} \cdot \frac{-2z}{(z^2 - 1)^2}\end{aligned}$$

5. Compute the following integrals.
   (a) $\int x^5 e^{-x^2}\,\mathrm{d}x$

   If we set $u = -x^2$, then $\mathrm{d}u = -2x\mathrm{d}$, or $-1/2\mathrm{d}u = x\mathrm{d}x$:

$$\int x^5 e^{-x^2}\,\mathrm{d}x = -\frac{1}{2}\int u^2 e^u\,\mathrm{d}u$$

To do the u-integral, we need to do integration by parts (twice), which is best done through tabular integration:

$+$	u^2	e^u
$-$	$2u$	e^u
$+$	2	e^u
	0	e^u

$$\int u^2 e^u \,\mathrm{d}u = u^2e^u - 2ue^u + 2e^u + \mathcal{C}$$

So therefore, in terms of our original variable x we have:

$$\int x^5 e^{-x^2}\,\mathrm{d}x = -\frac{1}{2}\left[x^4e^{-x^2} + 2x^2e^{-x^2} + 2e^{-x^2}\right] + \mathcal{C}$$

(b) $\displaystyle\int \frac{e^{3x}}{\sqrt{1-e^{6x}}}\,\mathrm{d}x$

We start with the substitution $u = e^{3x}$, so $1/3\mathrm{d}u = e^{3x}\mathrm{d}x$.

$$\begin{aligned}\int \frac{e^{3x}}{\sqrt{1-e^{6x}}}\,\mathrm{d}x &= \frac{1}{3}\int \frac{1}{\sqrt{1-u^2}}\,\mathrm{d}u\\ &= \frac{1}{3}\,\sin^{-1}(u) + \mathcal{C}\\ &= \frac{1}{3}\,\sin^{-1}(e^{3x}) + \mathcal{C}\end{aligned}$$

(c) $\displaystyle\int \frac{\cos(2x)}{1-\sin^2(2x)}\,\mathrm{d}x$

We start with the substitution $u = \sin(2x)$, so $1/2\,\mathrm{d}u = \cos(2x)\mathrm{d}x$.

$$\begin{aligned}\int \frac{\cos(2x)}{1-\sin^2(2x)}\,\mathrm{d}x &= \frac{1}{2}\int \frac{1}{1-u^2}\,\mathrm{d}u\\ &= \frac{1}{2}\,\tanh^{-1}(u) + \mathcal{C}\\ &= \frac{1}{2}\,\tanh^{-1}(\sin(x)) + \mathcal{C}\end{aligned}$$

(d) $\displaystyle\int x^3\sqrt{x^2-1}\,\mathrm{d}x$

First, we notice that we can rewrite the integrand as $x \cdot x^2\sqrt{x^2-1}$, and using the substitution $x = \sec(\theta)$ gives $\mathrm{d}x = \sec(\theta)\tan(\theta)\mathrm{d}\theta$. Using the identity $\sec^2(\theta) - 1 = \tan^2(\theta)$ and the substitution, the integral becomes

$$\begin{aligned}\int x^3\sqrt{x^2-1}\,\mathrm{d}x &= \int \sec^4(\theta)\tan^2(\theta)\,\mathrm{d}\theta\\ &= \int \sec^2(\theta)\tan^2(\theta)\,\sec^2(\theta)\mathrm{d}\theta\\ &= \int (1+\tan^2(\theta))\tan^2(\theta)\,\sec^2(\theta)\mathrm{d}\theta\end{aligned}$$

Using the substitution $u = \tan(\theta)$, $\mathrm{d}u = \sec^2(\theta)\mathrm{d}\theta$ we have yet another integral substitution:

$$\begin{aligned}\int (1+\tan^2(\theta))\tan^2(\theta)\,\sec^2(\theta)\mathrm{d}\theta &= \int (1+u^2)u^2\,\mathrm{d}u\\ &= \int u^2 + u^4\,\mathrm{d}u\\ &= \frac{1}{3}u^3 + \frac{1}{5}u^5 + \mathcal{C}\\ &= \frac{1}{3}\tan^3(\theta) + \frac{1}{5}\tan^5(\theta) + \mathcal{C}\end{aligned}$$

To get our integral back into terms of the original variable x, we recall that we originally had $x = \sec(\theta)$, which means $\cos(\theta) = 1/x$ and thus $\tan(\theta) = \sqrt{x^2-1}$. Thus

$$\int x^3\sqrt{x^2-1}\,\mathrm{d}x = \frac{1}{3}(\sqrt{x^2-1})^3 + \frac{1}{5}(\sqrt{x^2-1})^5 + \mathcal{C}$$

(e) $\displaystyle\int \cos^5(5x)\sin^7(5x)\,\mathrm{d}x$

We can let $u = \cos(5x)$ or $u = \sin(5x)$, either one works. To avoid dealing with a $-$ sign, we will choose $u = \sin(5x)$ with $1/5\,\mathrm{d}u = \cos(5x)\mathrm{d}x$. Using $\cos^2(5x) = 1 - \sin^2(5x)$, we have

$$\begin{aligned}
\int \cos^5(5x)\sin^7(5x)\,\mathrm{d}x &= \int (\cos^2(5x))^2\sin^7(5x)\ \cos(x)\mathrm{d}x \\
&= \frac{1}{5}\int (1-u^2)^2u^7\,\mathrm{d}u \\
&= \frac{1}{5}\int u^7 - 2u^9 + u^{11}\,\mathrm{d}u \\
&= \frac{1}{5}\left[\frac{1}{8}u^8 - \frac{2}{10}u^{10} + \frac{1}{12}u^{12}\right] + \mathcal{C} \\
&= \frac{1}{5}\left[\frac{1}{8}\sin^8(5x) - \frac{1}{5}\sin^{10}(5x) + \frac{1}{12}\sin^{12}(5x)\right] + \mathcal{C}
\end{aligned}$$